On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity and the case of $f(R)$-gravity

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in collaboration with Rituparno Goswami, Julien Larena, Peter Dunsby & Kishore Ananda, UCT
On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity

Talk based on

- **On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity**
  Naureen Goheer, Rituparno Goswami, Peter K. S. Dunsby, Kishore Ananda,

- **Power-law cosmic expansion in $f(R)$-gravity models**
  Naureen Goheer, Julien Larena, Peter K. S. Dunsby,

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Outline

- Why modify General Relativity?
- Why power-law solutions $a=t^m$?
- Reconstruct $f(G)$ such that it admits cosmologically interesting power-law
- Analogous Reconstruction for $f(R)$ gravity
- Conclude and discuss caveats
On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity

standard model of cosmology

- if we assume that the universe is dominated by Dark Energy (74%) and Dark Matter (22%), this model fits observational data very well
  - observations of CMB and LSS
  - keep GR and geometry

- shortcomings: dark matter and dark energy unexplained/ not observed directly

- $\Lambda$CDM model does not give theoretical explanation $\Rightarrow$ it is more of an empirical fit to data

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Standard Cosmology

**Key Assumptions:**

- General Relativity
- Homogeneity and Isotropy (FRW metric)
- Matter Content (Energy-Momentum tensor): dust+radiation

→ Decelerating universe in violation with observations

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Standard Cosmology

Accelerating universe

- change one (at least) of the key assumptions:
  - General Relativity
    - modify gravity on relevant scales
  - Homogeneity and Isotropy (FRW metric)
    - inhomogeneous models
  - Matter Content (Energy-Momentum tensor): dust+radiation
    - include more exotic fluids

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Standard Cosmology

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possible explanations of late time acceleration (DE)

\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \]

- change geometry (left hand side of field eqs)
- keep GR and modify “matter” content (change right hand side of field eqs)
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possible explanations of late time acceleration (DE)

\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \]

- modifications of cosm. thermodynamics (e.g. Chaplygin gas)
- add extra “fields” to 4D GR
  - cosmological constant
  - dynamical quintessence etc

change geometry (left hand side of field eqs)

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possible explanations of late time acceleration (DE)

\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \]

- inhomogeneous models
- modify GR on certain scales
  - higher D: brane-world models (Randall-Sundrum, DGP etc)
  - 4D: include higher order terms in action
- modifications of cosm.
  thermodynamics (e.g. Chaplygin gas)
- add extra "fields" to 4D GR
  - cosmological constant
  - dynamical quintessence etc

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modify GR by generalizing the action:

**Higher order gravity**

Generalize the Einstein-Hilbert action $A_{EH} = \int d^4x \sqrt{-g} R$
to include higher order curvature invariants:

$R \rightarrow$ function of Ricci scalar and contractions of Ricci tensor and Riemann
tensor: $f(R, R_{ab} R^{ab}, R_{abcd} R^{abcd}, etc)$

- abandon assumption of having second order field eqs (fourth order)
- unique status of GR was questioned by Weyl (1919) and Eddington (1922) by considering higher order invariants in the GR action
- higher order corrections to EH-action necessary to improve renormalizability of gravity (Utiyama, de Witt 1962 and more)

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Popular Higher order gravity models

Einstein-Hilbert action \( A_{EH} = \int d^4x \sqrt{-g}R \)

- replace \( R \) but some function \( f(R) \): \( A = \int dx^4 \sqrt{-g} f(R) \)

\( f(R) \)-gravity: very popular, good and relatively simple toy model; used in second part of this talk

- include functions of the Gauss-Bonnet curvature invariant \( G \):
  \[
  A = \int d^4x \sqrt{-g}[R + f(G)], \quad G = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}
  \]

\( f(G) \)-gravity (main objective here)

- other options?!

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**Popular Higher order gravity models**

Einstein-Hilbert action \[ A_{EH} = \int d^4 x \sqrt{-g} R \]

- **replace $R$ but some function $f(R)$**: \[ A = \int dx^4 \sqrt{-g} f(R) \]

  \[ f(R) \text{-gravity}: \text{very popular, good and relatively simple toy model; used in second part of this talk} \]

- **include functions of the Gauss-Bonnet curvature invariant $G$**: \[ A = \int d^4 x \sqrt{-g} (R + f(G)), \quad G = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \]

  \[ f(G) \text{-gravity} \text{ (main objective here)} \]

- **other options?!**

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Simple example to show mechanism:

\[ f(R) = R - \frac{\mu^4}{R} \]

- mass scale \( \mu \propto H_0 \propto 10^{-33} \text{ eV} \)

- high curvature \((R \gg 1)\): 1/R correction **negligible**

- low curvature \((R \rightarrow 0)\): 1/R correction kicks in and gives **late time acceleration**

- ruled out (weak field limit and Dolgov-Kawasaki instability)!

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Requirements for viability

- correct cosmological dynamics (correct expansion history radiation domination -> matter domination -> late time acceleration)

- correct Newtonian / post-newtonian limit (solar system constraints)

- correct perturbations: agree with CMB and LSS

- agree with BBN

- no ghosts / instabilities/ (curvature) singularities

  - model dependent (frame dependent!), but there seem to be successful candidates in both $f(R)$ - and $f(G)$ -gravity

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On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity

Basic Idea

• look at **various classes of** $f(R)$ - and $f(G)$ - gravity that are popular in literature because they are claimed to be **viable models** (agree with observations, no instabilities, ...)

• construct **dynamical systems** state space and identify **equilibrium points** as physically interesting states of the system

• could **not find equilibrium points corresponding to matter dominated power-law solutions** in various classes of models
Dynamical System example:
FRW state space in GR, perfect fluid, \( \Lambda, k=\pm 1 \)

\[ \Omega_k = \frac{3R}{\mathcal{H}^2} \]

\[ \Omega_\Lambda = \frac{\Lambda}{\mathcal{H}^2} \]
Einstein Static models if f(R)-gravity

Dynamical System example:
FRW state space in GR, perfect fluid, $\Lambda$, $k=\pm 1$

\[
\Omega = 0
\]

\[
\Omega_\Lambda = \frac{\Lambda}{H^2}
\]

\[
\Omega_k = \frac{3R}{H^2}
\]

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Dynamical System example:
FRW state space in GR, perfect fluid, $\Lambda$, $k=\pm 1$

- Friedman model $a=t^{2/3(1+w)}$: past attractor

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Dynamical System example:
FRW state space in GR, perfect fluid, $\Lambda$, $k=\pm 1$

- Friedman model $a=t^{2/(3(1+w))}$: past attractor
- deSitter model $a=e^{\Lambda t}$: future attractor

$\Omega_k = \frac{3R}{H^2}$

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Einstein Static models if f(R)-gravity

Dynamical System example:
FRW state space in GR, perfect fluid, $\Lambda$, $k=\pm 1$

- Friedman model $a=t^{2/3(1+w)}$: past attractor
- deSitter model $a=e^{\Lambda t}$: future attractor
- integrate perturbation eqs along trajectory

$\Omega_k = \frac{3R}{H^2}$

$\Omega_\Lambda = \frac{\Lambda}{H^2}$

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why power-law solutions?

- at early times, the universe was radiation and matter dominated, and the scale factor scaled like $a = a_0 \, t^{1/2}$, $a = a_0 \, t^{2/3}$

- in GR, there exists the exact power-law solution $a = a_0 \, t^{2/3(1+w)}$, which then justifies looking at the perturbed solution $\rightarrow$ structure growth

- if there is no such asymptotic solution in the modified gravity of interest, one has to be very careful about which background to consider for perturbation theory

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## Simple Example: $R^n$-gravity

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| $D$   | $[2(1-n), 2(n-1)^2, 0]$ | $\begin{cases} 
a = \frac{kt}{2n^2-2n-1} & \text{if } k \neq 0 \\
0 & \text{if } k = 0
\end{cases}$ |
| $E$   | $[-1 - 3\omega, 0, -1 - 3\omega]$ | $a = a_0(t - t_0)$ |
| $F$   | $[1 - 3\omega, 0, 2 - 3\omega]$ | $a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$) |
| $G$   | $\left[-\frac{3(n-1)(1+\omega)}{n}, \frac{(n-1)[4n-3(\omega+1)]}{2n^2}, \frac{n(13+9\omega)-2n^2(4+3\omega)-3(1+\omega)}{2n^2}\right]$ | $a = a_0 \left(\frac{2n}{3(1+\omega)}\right)$ |

- **evolve density perturbations for exact solution (or along the orbit)**
- **generalize to other $f(R), f(G)$**
Basic Idea

- could **not find equilibrium points** in the dynamical systems corresponding to **matter dominated power-law models** in various classes of $f(R)$- and $f(G)$-gravity
  - these points represent **asymptotic/intermediate states** in the full state space of possible cosmological evolutions (--- solutions interpolating between matter domination and acceleration)
  - important as **backgrounds for perturbations**

- go **back to the basic field equations** and assume there exists a power-law solution
  - in each case, ones gets very strong **restrictions** on the form of $f$
  - in $f(G)$-gravity there is an **additional complication** that may make this type of modified gravity less attractive than $f(R)$-gravity

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why $f(G)$-gravity?

- motivated by low-energy effective string theory (Nojiri & Odintsov 2006)?
  - ST ‘naturally predicts occurrence of terms with inverse powers in curvature invariants in low energy effective action’
  - in ‘string induced gravity’, terms of the Gauss-Bonnet term $G$ coupled to a scalar field $\phi$ in the action may help obtain a non-singular cosmology
  - dropping kinetic term in action can be interpreted as $f(G)$ in action

- can possibly give late-time acceleration without cosmological constant and simultaneously pass solar system constraints (might be less constrained by local gravity tests compared to $f(R)$, Sotiriou 2007)

- ghosts may be avoided under certain conditions (De Felice et al 2006)

- could be promising candidates for cosmology, study expansion history!

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$f(G)$ field equations

- **Action** for 4-dimensional **homogeneous isotropic** backgrounds:

\[ A = \int d^4x \sqrt{-g} \left[ R + f(G) + \mathcal{L}_m \right], \quad G = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \]

- we know that in 4 dimensions, $G$ is a total differential, and field equations are invariant if we add terms linear in $G$

- **varying with respect to the metric** gives

\[ G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa^2 \left( T^m_{\alpha\beta} + T^G_{\alpha\beta} \right), \text{ where } T^M_{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\alpha\beta}} \]

- treat fourth order gravity as **GR with 2 effective fluids**!

- unlike in $f(R)$, standard matter decouples from curvature corrections!

- curvature “fluid” moving relative to standard matter, no perfect fluid!:

\[ T^G_{\mu\nu} = -8 \left[ R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\mu\nu} - R_{\mu\nu}g_{\rho\sigma} + R_{\mu\sigma}g_{\nu\rho} \right] \]

\[ \quad + \frac{R}{2} \left( g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho} \right) \nabla^\rho \nabla^\sigma G + (f - Gf_G) g_{\mu\nu}, \]

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$f(G)$ field equations for FRW backgrounds

As usual, define the **scale factor** (volume expansion) $\Theta = \frac{\dot{a}}{a} = 3H$

- **Raychaudhuri**: $\dot{\Theta} + \frac{1}{3} \Theta^2 = -\frac{\kappa^2}{2} (\rho + 3P) + \frac{4}{9} \Theta^3 f_{g\dot{g}} \dot{G} - (f - G f_g) - \frac{3}{\Theta} g \dot{f}_g - \frac{4}{3} \Theta^2 \ddot{f}_g$

- **Friedman**: $\Theta^2 = 3\kappa^2 \rho + 3G f_g - \frac{8}{3} \Theta^3 f_{g\dot{g}} \dot{G} - 3f$

- **total trace**: $-4\Theta^2 - 6\dot{\Theta} = 3\kappa^2 (3P - \rho) + 12 (f - G f_g) + \frac{8}{3} \Theta^3 f_{g\dot{g}} \dot{G} + \frac{18}{\Theta} G \dot{f}_g + 8\Theta^2 \ddot{f}_g$

where for n=1,2 we abbreviate $f_{ng} = \frac{\partial^n f}{(\partial G)^n}$

- **energy conservation** eq. for standard matter: $\dot{\rho} = -\Theta (\rho + P)$

- for FRW spacetimes, the GB-term becomes: $\mathcal{G} = \frac{8}{9} \Theta^2 \left[ \dot{\Theta} + \frac{1}{3} \Theta^2 \right] = 24 \frac{\dot{a}^2 \ddot{a}}{a^3}$

  - note that $\mathcal{G} < 0$ for **deceleration** and $\mathcal{G} > 0$ for **acceleration**

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Requirements for the existence of power-law solutions

• assume there exists an **exact** power-law (PL) solution \( a = a_0 t^m \)

• \( m \) is from now on fixed;
  
  ‣ \( 0 < m < 1 \): the PL solution is **decelerating**
  ‣ \( m > 1 \): the PL solution is **accelerating**

• from energy conservation we get \( \rho = \rho_0 \, t^{-3m(1+w)} \)

• Gauss-Bonnet term becomes \( G = 24 \, m^2 \, (m-1) \, t^{-4} = \alpha_m \, t^{-4} \)
  
  ‣ \( G < 0 \): the PL solution is **decelerating**
  ‣ \( G > 0 \): the PL solution is **accelerating**

• can **invert** GB term to **get** \( t \) as a function of \( G \) (assume \( t > 0 \))
If power-law solution exists we can...

- insert PL solution $a = a_0 t^m$ into the 3 independent field equations

- replace time $t$ by $G$ to get linear differential eqs for $f(G)$ in $G$-space; the Friedman eq. e.g. becomes

$$
\frac{96m^3}{\alpha_m} f_{gg} G^2 + f_g G - f - 3m^2 \sqrt{\frac{G}{\alpha_m}} + K \left( \frac{G}{\alpha_m} \right)^{\frac{3}{4}} m(1+w) = 0,
$$

where $K = \rho_0 a_0^{3(1+w)}$, $\alpha_m = 24m^3(m-1)$

- general solution:

$$
f(G) = A_m \sqrt{\tilde{G}} + B_{mw} \tilde{G}^{\frac{3}{4}} m(1+w) + C_1 G + C_2 G^{\frac{1}{4}} - \frac{m}{4}, \quad \tilde{G} = 24G/\alpha_m
$$

  - in particular, many of the popular $f(G)$ models (e.g. de Felice et al, 2009) cannot admit any exact PL solution

- set $C_1=C_2=0$ because of GB-invariance and to get GR limit respectively

- $A_m, B_{mw}$ real-valued and non-zero unless $m=1$

- this solution satisfies the other field equations!

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Co-existence of decelerating PL solution and ANY accelerating solutions

• assume an **exact decelerating PL solution exists**
  - $m < 1$, $\alpha_m < 0$ **fixed** numbers

• from above we know that then $f(G)$ HAS TO BE of the form
  $$f(G) = A_m \sqrt{\tilde{G}} + B_{mw} \tilde{G}^{\frac{3}{4}m(1+w)} \cdot \tilde{G} \equiv 24G/\alpha_m > 0$$

• if any **additional accelerating** solution exists, it has by definition $G > 0$
  - $f(G)$ not real-valued!!

• introduce absolute values to “fix” problem: $\tilde{f}(G) = A_m \sqrt{|\tilde{G}|} + B_{mw} |\tilde{G}|^{\frac{3}{4}m(1+w)}$
  - function **not differentiable** at $G=0 \rightarrow$ not $C^2$ as required for action

• no $C^2$-action in $f(G)$-gravity can allow for exact decelerating power-law solution to coexist with ANY accelerating solution

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The $f(R)$ analogue

• generalize the Einstein-Hilbert action to

\[ A = \int dx^4 \sqrt{-g} \, f(R) + \int dx^4 \sqrt{-g} \, \mathcal{L}_m \]

• relatively simple, but has the nice feature of admitting late time accelerating models (alternative to DE)

• may or may not comply with observational constraints (model dependent, some successful models at the least)

• **varying with respect to the metric** gives the field equations:

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The $f(R)$ analogue: Basic field equations

- Raychaudhuri:  
  \[ \dot{\Theta} + \frac{1}{3} \Theta^2 = -\frac{1}{2f'} \left[ \rho + 3P + f - f'R + \Theta f'' \ddot{R} + 3f''' \dot{R}^2 + 3f'' \dddot{R} \right] \]

- Friedman:  
  \[ \Theta^2 = \frac{3}{f'} \left[ \rho + \frac{R f' - f}{2} - \Theta f'' \dot{R} \right] \]

- total trace:  
  \[ 3\ddot{R} f'' = \rho - 3P + f' \dot{R} - 2f - 3\Theta f'' \ddot{R} - 3f''' \dot{R}^2 \]

where a prime denotes a derivative with respect to $R$

- energy conservation:  
  \[ \dot{\rho} = -\Theta (\rho + P) \] both acceleration and deceleration can occur for $R > 0$

- combine above:  
  \[ R = 2\dot{\Theta} + \frac{4}{3} \Theta^2 \]

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Requirements for the existence of exact power-law solutions in $f(R)$

- assume again that there exists an exact power-law (PL) solution $a = a_0 t^m$
  - $0 < m < 1$: the PL solution is decelerating
  - $m > 1$: the PL solution is accelerating

- as in the $f(G)$ case, the energy conservation yields $\rho = \rho_0 t^{-3m(1+w)}$

- instead of Gauss-Bonnet term we get $R = 6m (2m-1) t^2 = \alpha_m t^{-2}$
  - Unlike in the $f(G)$ analogue, both accelerating and decelerating solutions can have $R > 0$ (as long as $m > 1/2$).

- invert $R$ to get $t$ as a function of $R$ (assume $t > 0$), insert PL solution $a = a_0 t^m$ into the 3 independent field equations and replace time $t$ by $R$ to get linear differential eqs for $f(R)$ in $R$-space

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Requirements for the existence of exact power-law solutions in $f(R)$

- the **Friedman** eq. e.g. becomes

$$f''R^2 + \frac{m - 1}{2} f'R + \frac{1 - 2m}{2} f + (2m - 1)K \left(\frac{R}{\alpha_m}\right)^{\frac{3}{2}m(1+w)} = 0,$$

where $K = \rho_0\alpha_0^{3(1+w)}$, $\alpha_m = 6m(2m - 1)$

- **general solution:**

$$f(R) = A_{mw} \left(\frac{R}{\alpha_m}\right)^{\frac{3}{2}m(1+w)} + C_1 R^{\frac{3}{4} - \frac{m}{4} + \frac{\sqrt{\beta_m}}{4}} + \frac{2}{\sqrt{\beta_m}} C_2 R^{\frac{3}{4} - \frac{m}{4} - \frac{\sqrt{\beta_m}}{4}},$$

where $\beta_m = 1 + 10m + m^2$

- $A_{mw}$ **real-valued and non-zero** unless $m=1/2$ or $w = \frac{3 - 7m \pm \sqrt{\beta_m}}{6m}$

- this solution **satisfies the other field equations**!

- if we want GR limit for $m=2/(3(1+w))$ we have to set $C_1=C_2=0$

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this means...

- **exact** power-law solution solution only exists for

$$f(R) = A m w \left( \frac{R}{\alpha m} \right)^{ \frac{3}{2} m (1+w) } + C_1 R^{ \frac{3}{4} - \frac{m}{4} + \frac{\sqrt{\beta_m}}{4} } + \frac{2}{\sqrt{\beta_m}} C_2 R^{ \frac{3}{4} - \frac{m}{4} - \frac{\sqrt{\beta_m}}{4} },$$

where $\beta_m = 1 + 10m + m^2$

- actions including terms such as $R + \alpha F(R)$ (e.g. $f(R)=R+R/(1+A R)$ as studied by Hu et al) cannot have **exact** power-law solutions

- re-write the solution for $f(R)$ as

$$f(R) = B m w R^n, \text{ where } n \equiv \frac{3}{2} m (1+w), \quad C_1 = C_2 = 0$$

  - recover the well-known result that $R^n$-**gravity** allows for an **exact** Friedman like power-law solution $a=a_0 t^{2n/(3(1+w))}$

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Simple Example: $R^n$-gravity

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<tr>
<td>$E$</td>
<td>$[-1 - 3\omega, 0, -1 - 3\omega]$</td>
<td>$a = a_0(t - t_0)$</td>
</tr>
<tr>
<td>$F$</td>
<td>$[1 - 3\omega, 0, 2 - 3\omega]$</td>
<td>$a = a_0(t - t_0)^{1/2}$ (only for $n = 3/2$)</td>
</tr>
<tr>
<td>$G$</td>
<td>$\left[-\frac{3(n-1)(1+\omega)}{n}, \frac{(n-1)(4n-3(\omega+1))}{2n^2}, \frac{n(13+9\omega)-2n^2(4+3\omega)-3(1+\omega)}{2n^2}\right]$</td>
<td>$a = a_0 t^{2n/(3(1+\omega))}$</td>
</tr>
</tbody>
</table>

$\implies$ evolve density perturbations for exact solution (or along the orbit)

$\implies$ generalize to other $f(R), f(G)$

$S = S_0 t^{\frac{(1-n)(2n-1)}{n-2}}$

$1.36 < n < 1.5$

$S = S_0 t^{\frac{2n}{3(1+w)}}$
Comparison between $f(G)$- and $f(R)$-gravity

\[ G = \frac{8}{9} \Theta^2 \left[ \dot{\Theta} + \frac{1}{3} \Theta^2 \right] = 24 \frac{\dot{a}^2 \ddot{a}}{a^3} \]

• models transitioning from deceleration to acceleration must pass through $G=0$
  ▸ functions like $f(G) = G^n (n<1)$ problematic for cosmologically viable trajectories

\[ R = 2 \dot{\Theta} + \frac{4}{3} \Theta^2 \]

• they must not pass through $R=0$
  ▸ function like $f(R) = R^n (n<1)$ well-defined for at least some cosmologically viable trajectories

• example $R^n$:
  ▸ matter dominated phase coexists with dS-like solution, and can be linked without passing through $R=0$
  ▸ not differentiable at $R=0$, but any solution can only asymptotically approach this plane from either side of 0

Naureen Goheer, University of Cape Town
On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity

Scalar field analogy for $f(R)$ case

- **reconstruct the effective scalar field** (often done to describe the dynamics of $f(R)$ gravity models)

- has been argued that $f(R)$ theories suffer from a singularity problem (Frolov 2008):
  - **at finite time** the dynamics drives the model towards **infinite values of the curvature** corresponding to points in the scalar field potential attainable for finite values of the scalar field
  - effective potential of the models are **multi-valued** (very unnatural feature)

- we will show that the models that lead to power-law solutions do **not suffer from such pathological behaviors**, but admit a well-defined scalar field representation with a single-valued potential and no curvature singularity.

Naureen Goheer, University of Cape Town
On the co-existence of matter dominated and accelerating solutions in $f(G)$-gravity

**Scalar field analogy**

- adopt the scalar field representation defined by

$$
\phi = \frac{df(R)}{dR} - 1 ,
$$

$$
\frac{dV}{dR} = \frac{1}{3} \left( 2f(R) - \frac{df}{dR} R \right) \frac{d^2 f}{dR^2} .
$$

- can obtain scalar field potential for the $f(R)$ that has exact power-law solution as shown above:

$$
f(R) = A_{mw} \left( \frac{R}{\alpha_m} \right)^{\frac{3}{2}m(1+w)} + C_1 R^{\frac{3}{4} - \frac{m}{4} + \frac{\sqrt{\beta_m}}{4}} + \frac{2}{\sqrt{\beta_m}} C_2 R^{\frac{3}{4} - \frac{m}{4} - \frac{\sqrt{\beta_m}}{4}} ,
$$

where $\beta_m = 1 + 10m + m^2$

Naureen Goheer, University of Cape Town
Reconstructed scalar field potential

\[ f(R) = A_{m,w} \left( \frac{R}{\alpha_m} \right)^{\frac{3}{2} m(1+w)} + C_1 R^{\frac{3}{4} - \frac{m}{2} + \sqrt{\frac{\sigma_m}{4}}} + \frac{2}{\sqrt{\beta_m}} C_2 R^{\frac{3}{4} - \frac{m}{2} - \sqrt{\frac{\sigma_m}{4}}}, \]

where \( \beta_m = 1 + 10m + m^2 \)

- \( n = \frac{3m(1+w)}{2} > 1 \)
  - shape of potential does not depend on \( C_1, C_2 \) and \( w \)
  - scalar field rolls down potential and asymptotically freezes at \( \phi = -1 \)
- \( n = \frac{3m(1+w)}{2} < 1 \)
  - shape of potential depends on \( C_1, C_2 \), but scalar field still drives \( V \) towards constant value at late times

Naureen Goheer, University of Cape Town
Scalar field analogy

- reconstructed scalar field well-behaved

  - effective potential single-valued
  - no curvature singularity
  - scalar field potential asymptotically freezes at finite value
Conclusion

• in both $f(G)$- and $f(R)$-gravity, the existence of an exact power-law solution restricts the function $f$ to one class of models
  ‣ same is true for other simple forms of scale factor evolution $a(t)$

• in $f(G)$-gravity, an exact decelerating power-law solution can not co-exist with ANY accelerating solutions for a $C^2$-function $f(G)$
  ‣ less promising than $f(R)$ models for cosmologically viable expansion histories?
  ‣ concentrate on functions $f(G)$ that are $C^2$ (and in particular differentiable at $G=0$) or ignore the level of the action?

⇒ the requirement for an exact PL solution to exist may be too stringent?
  ‣ in $\Lambda$CDM, matter dominated $a=t^{2/3}$ exact solution for $\Lambda<<H^2$
  ‣ numerically, matter dominated phases were found? approximations? generic?

⇒ provide a formalism to reconstruct the function $f$ given a simple expansion history

⇒ in $f(R)$-gravity, $R^n$-gravity is very special: generically allows for exact power-law solutions in the non-GR limit (even for e.g. $n=2$)
Matter Power spectrum in $R^n$-gravity (Ananda et al)

- On **large** and **small** scales the spectrum is scale-invariant, but **Oscillations** can occur around a specific value of $k$ depending on parameter “$n$”.

- in contrast to results obtained using **approximation** schemes (**no exact solutions**): increased power at small scales and no oscillations

- inconsistencies with exact formalism, or $R^n$ just very special? $\rightarrow$ need to test approximation schemes!
On large and small scales the spectrum is scale-invariant, but Oscillations can occur around a specific value of $k$ depending on parameter “$n$”.

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On large and small scales the spectrum is scale-invariant, but Oscillations can occur around a specific value of k depending on parameter “n”.

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On large and small scales the spectrum is scale-invariant, but Oscillations can occur around a specific value of $k$ depending on parameter “$n$”. 

in contrast to results obtained using approximation schemes (no exact solutions): increased power at small scales and no oscillations

inconsistencies with exact formalism, or \( R^n \) just very special? -> need to test approximation schemes!
Effect of fourth order gravity evident only around a specific value of $k$.

- On large and small scales the spectrum is scale-invariant, but Oscillations can occur around a specific value of $k$ depending on parameter “$n$”.

- In contrast to results obtained using approximation schemes (no exact solutions): increased power at small scales and no oscillations.

- Inconsistencies with exact formalism, or $R^n$ just very special? -> need to test approximation schemes!
Thank you!
Caveats

- we have assumed there exists an **exact decelerating PL solution**

  - if \( a = a_0 e^{nt} t^m \) or \( a = a_0 t^m + a_1 t^n \) then at early times \( a = a_0 e^{nt} t^m \) but the basic assumption of our analysis is not satisfied

  - in this case, \( f \) could be of a much more complicated form, and in the limit of early or late times, \( f \) may or may not (non-linearities) scale like the class of \( f \) obtained above
The Power Spectrum for $R^n$-gravity ($n>1$)
The Power Spectrum for $R^n$-gravity ($n>1$)